Selection by pairwise comparisons with limited resources

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Abstract

We analyze different methods of sorting and selecting a set of objects by their intrinsic value, via pairwise comparisons whose outcome is uncertain. After discussing the limits of repeated Round Robins, two new methods are presented: The *ran-fil* requires no previous knowledge on the set under consideration, yet displaying good performances even in the least favorable case. The *min-ent* method sets a benchmark for optimal dynamic tournaments design.

Key words: Tournament design, information filtering

Introduction

In the Internet era the amount of available information is overwhelming: the problem of finding and selecting the most relevant becomes, therefore, crucial. Each selection operation is noisy, yet we want to estimate the minimal amount of resources that is necessary to sort a large number of items. Our study is a first step to establish a firm theoretical bound for the new Information Theory introduced in [1], which aims to give theoretical basis for evaluating and improving the current and future search engines.

The method of paired comparisons has been extensively studied by statisticians [2] and widely applied in various fields. Usually one has to rank N objects (or agents), each one of them endowed with a scalar quality $q_i, i = 1, 2, ..., N$, on the basis of a finite number of binary comparisons t_c. The "true" rank R_i of item i is assumed to be zero if q_i is the highest quality, one for the second highest and so forth. Upon assuming an *a priori* probability distribution of quality $\phi(q)$ and a given probability distribution of outcomes of comparisons between two objects of known quality $p_{i,j}$, it is possible to write the joint likelihood of the outcomes as a function of individual qualities. The best guess for the quality set is the one that maximizes the likelihood, i.e. the one that would produce the given outcome with the highest probability.

On the other hand scarce attention has been devoted so far to the problem of designing optimal tournaments with a limited number of games. Round Robin (RR) tournaments, for instance, are common but not very effective, because they assume that all comparisons are equally useful. In fact one might be more interested in the upper part of the classment than in its lower end and the result of some comparisons could be foreseen with high precision on the basis of previous outcomes. This fact motivates the filters we shall develop in the present paper. Let us first discuss the theoretical limits of repeated RRs with the following example.

1 Best selection via Round Robins

We want to select the best out of $N = N(0)$ objects, by successive elimination, in k_c rounds. Round k is completed once a RR among all the (surviving) $N(k)$ objects is performed; after each round $\Delta(k)$ objects are eliminated. Thus

$$N(k+1) = N(k) - \Delta N(k).$$

If the elimination were perfectly effective, after k rounds the quality of the selected objects would be uniformly distributed in the range $(1 - \gamma(k), 1)$, where $\gamma(k) \propto N(k)/N$.

Consider now two objects i and j, with a difference in mutual preference probabilities $\epsilon_{i,j} = |p_{ij} - p_{ji}|$, that are compared ν times. Their average difference in points will be of order $\nu\epsilon \pm \sqrt{\nu}$. In many models of interest, like the Bradley Terry (BT) model introduced below (4), the average ϵ of $\epsilon_{i,j}$ in the quality domain $(1-\gamma, 1)$ is proportional to γ. In this case, in order to have a significant separation of two agents, they have to play at least

$$\nu \sim 1/\epsilon^2(k) \propto 1/\gamma^2(k) \propto N^2/N^2(k) \tag{1}$$

games on average. In addition to that, we should keep in mind that each object is compared with the remaining $N-1$ in the first round. If we want to keep a constant selecting power, another factor $N/N(k)$ has then to be considered to account for the diminishing number of opponents. In all, each object needs be compared $\nu \sim N^3/N^3(k)$ times before deciding if it survived round k.

Now suppose that, at round zero, we eliminated a percent of the original N

objects. At round k we shall eliminate the same a percent of the remaining $N(k)$ objects only in about ν rounds. Thus

$$\Delta N(k) \propto a \left[\frac{N(k)}{N}\right]^3$$

If we take the continuum limit and solve the resulting differential equation, this yields

$$N(t) = 2aN/\sqrt{t}. \tag{2}$$

The minimum number n_c of comparisons needed for the champion to arise can be calculated by integrating (2) from 0 to $k_c \propto N^2$. The result reads

$$n_c \propto N^2 \log N. \tag{3}$$

In the remaining of this paper we will propose two new procedures intended to improve this result.

2 The ran-fil method

We shall introduce an algorithm that can be used for finding an object with a large quality, i.e. a highly ranked one, with no previous knowledge of $\phi(q)$ and $p_{i,j}$. Inspired by a previous work [1], it is constructed in terms of rounds. In each round we line up objects labeled with numbers $1, \ldots, N$ (N assumed to be even) and compare all the odd numbered ones with their successors of even numbers: 1 with 2, 3 with 4 and so on. A round ends once items $N-1$ and N have been compared. Winners replace losers: if the object at site i is preferred over that at $i+1$, we replace the latter with i, and vice versa. Before a new round starts all objects are reshuffled. We define the time t as the number of pairwise comparisons made (excluding cases where an agent is compared to himself). Table 1 shows an example of a run of our algorithm where, after k rounds, q_6 is our designed winner, i.e. our guess for the agent with the best quality.

Using the above approach, however, there is a quite large probability of losing the item with the highest quality in one of the initial rounds and therefore end up with a poor average winner's rank R_w. In order to avoid this we introduce some noise such that, in each round we reintroduce objects which have been eliminated in earlier rounds. The noise level we use is fixed and set by the number of agents η we re-introduce in each round. Agents gain one point for each time they have been preferred. Finally we estimate the agent with the best rank, for a given time step t_c, as the one who gained more points.

n	12	34	56	78
start	q_1q_2	q_3q_4	q_5q_6	q_7q_8
1	q_2q_2	q_3q_3	q_6q_6	q_8q_8
shuffle	q_6q_3	q_3q_8	q_8q_2	q_2q_6
2	q_6q_6	q_3q_3	q_2q_2	q_6q_6
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
k	q_6q_6	q_6q_6	q_6q_6	q_6q_6

Table 1
Example of a run of the ran-fil algorithm with no noise.

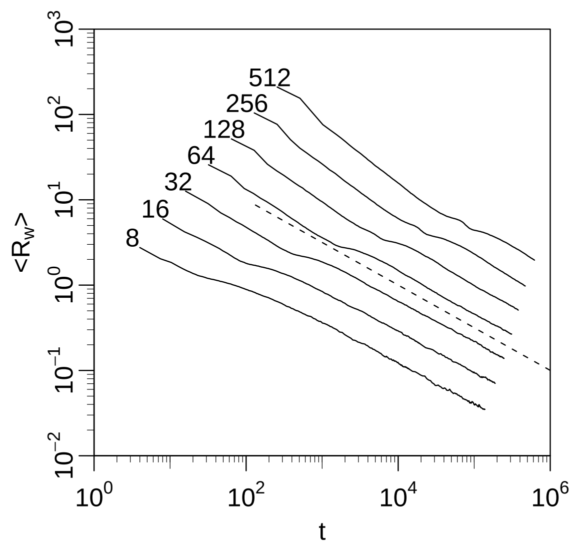

Fig. 1. The average rank $\langle R_w \rangle$ of the winning agent versus the number of comparisons in the run-filter method, for various values of N (shown next to each line). The rank is averaged over 10000 runs and the noise level is put such that every second round an agent is re-introduced, i.e. $\eta = .5$. The dashed line is proportional to $1/\sqrt{t}$.

2.1 Testing the run-filter on the BT model

Let us now focus on the model

$$p_{i,j} = \frac{q_i}{q_i + q_j}, \tag{4}$$

first proposed by Zermelo [4] but often referred to as Bradley-Terry (BT) model [5]. We tested the run-filter method on a population of agents whose qualities were uniformly distributed between zero and one. Fig. 1 shows the average rank of the winner as a function of the total number of comparisons t_c, for different values of N and with a reintroduction level $\eta = 0.5$. For the first few rounds our algorithm is similar to a knock-out tournament, i.e. the number of teams in each round is halved. Note that the ranking after some transient shows a characteristic scaling in the number of comparisons $t_c \propto R_w^\beta$, and that the scaling exponent β seems to be independent of the number of agents.

In the BT model the difference in mutual preference probabilities of two agents with qualities α and $\alpha - \delta$ ($1 \gg \delta > 0$) is

$$\epsilon = \frac{\alpha}{2\alpha + \delta} - \frac{\alpha - \delta}{2\alpha + \delta} = \frac{\delta}{2\alpha} + O(\delta^2). \tag{5}$$

This means that when the surviving agents all have qualities close to one, their average difference in ϵ scales almost linearly with δ and equation (1) holds. In the large N limit we can thus write down a differential equation for $\epsilon(t)$:

$$\frac{d\epsilon}{dt} = -\alpha\epsilon^3, \tag{6}$$

where α is a constant. The solution is explicitly written as

$$\epsilon(t) = \frac{1}{\sqrt{2\alpha}} t^{-1/2}. \tag{7}$$

It now follows that the estimated rank, $R_w(t)$, can be approximated by

$$R_w(t) \approx \frac{N}{\epsilon(t)} \int_{1-\epsilon(t)}^{1} (1 - q) dq = \frac{1}{\sqrt{8\alpha}} t^{-1/2}.$$

In fig. 1 we have added a line with a scaling exponent of $\beta = -0.5$ in the number of comparisons. The numerical examples are after some transient in excellent agreement with the predicted scaling (7).

The ran-fil algorithm is directly applicable when you want to estimate the complete ranking table. In this case we define ϵ as

$$\epsilon(t) = \frac{1}{N} \sum_{i=1}^{N} |\tilde{R}_i(t) - R_i|,$$

where $\tilde{R}_i(t)$ is the estimated rank of agent i at time t and R_i its true rank. Following the same argument as above we get the same scaling exponent. The prefactor, however, is larger.

Note that the noise is defined as simple as possible without any reference to an underlying distribution of $p'_{ij}s$. One could, based on *a priori* knowledge of such a distribution, tune the noise to increase the performance of our filter. Below we show that tuning of the noise can improve performance dramatically.

3 The min-ent method

We shall outline here a method in which we assume that the fitness distribution $\phi(q)$ and the functional form of $p_{i,j} = f(q_i, q_j)$ are known a-priori. In real life it is very rarely the case, unless these quantities can be reliably estimated from data collected in the past. In some sports, for example, previous championships could provide such data. Further inquiry is needed to check how robust this method is with respect to errors in the existing knowledge. Similar problems are widely dealt with in the literature and so we shall not tackle this question here.

The idea of our method is to choose the couples to compare dynamically during the tournament. At every time step a comparison $x(t)$ is performed and its outcome $w(t)$ recorded. By time step t each couple (i, j) has played $n_{i,j}(t) \in (0, t)$ games. Let us denote by $w_{i,j}(t)$ the number of times i has beaten j at time t, and by $W(t) = ((w_{i,j}))$ the matrix of results collected until time t. It follows that $w_{i,j}(t) + w_{j,i}(t) = n_{i,j}(t)$ for each pair.

Let us now focus on the problem of finding only the best item. Without any prior information it is natural to draw the couples to compare in the first round from a uniform distribution. Once matrix $W(t)$ is connected, though, we can adapt such a distribution so as to maximize the acquisition of new information. The information provided by a new comparison $x_{u,v}$ can be quantified as

$$I_{W(t)}(x) = 1 - H\left(\psi|W(t) \cup x_{u,v}\right)/H_{max}, \tag{8}$$

where $H(\psi|W(t) \cup x_{u,v})$ is the entropy of the conditional distribution $\psi(k|W(t) \cup x_{u,v})$, the probability that site k has the highest fitness, given the matrix of outcomes till time t plus a new comparison $x_{u,v}$. Ideally one would like such a probability distribution to be as picked as possible around the most probable value, which translates into information maximization. This is a general procedure, but different models require specific definitions of the expected conditional information and of the weights characterizing the importance sampling one wishes to apply [6].

Here we shall test the procedure of comparing the maximizing couple at each time step, i.e.

$$x(t+1) = Arg \max_{x_{u,v}} I_{W(t)}(x) \tag{9}$$

with the following definition of ψ:

$$\psi(k|W(t) \cup x_{u,v}) = p_{u,v}\psi(k|W(t) \cup w_{u,v}) + p_{v,u}\psi(k|W(t) \cup w_{v,u}).$$

This corresponds to taking the expected value of unknown outcomes. Extension to the determination of the entire rank or part of it can be easily found with the same reasoning.

3.1 Finding the best item in a test model

We test the outlined method on the model [7]

$$\begin{aligned} p_{b,j} &= \pi > 0, j \neq b \\ p_{i,j} &= 0.5, i, j = 1, 2, ..., N; i, j \neq b, \end{aligned} \tag{10}$$

which is the least favorable among many instances [9] and analytically solvable in the case of Round Robins [2]. Here, clearly, the assumption on a-priori distributions translates into known win probabilities. Thus

$$\psi(k|W(t)) \propto \pi^{w_k}(1-\pi)^{n_k - w_k},$$

where $n_k = \sum_j n_{k,j}$ and $w_k = \sum_j w_{k,j}$.

We proceed as follows: at least N comparisons are previously made in such a way that the matrix of outcomes be connected. Then, at each time step t, we compute, for all possible couples (u, v), the conditional entropy

$$H\left(\psi|W(t) \cup x_{u,v}\right) = \frac{H(\psi|W(t)) + (1-a_\pi)(\psi_u \log \psi_u + \psi_v \log \psi_v) - a_\pi \log a_\pi(\psi_u + \psi_v)}{(1 - (\psi_u + \psi_v)(1 - a_\pi))}, \tag{11}$$

where $a_\pi = 2\left[1 - 2\pi(1-\pi)\right]$ and ψ_y stands for $\psi(y|W(t))$. Next time step we shall compare the couple $x(t+1)$ satisfying condition (9).

We assign $\sum_{j \neq k} w_{k,j}/n_{k,j}$, points to item k and declare the winner as the one that collected more points at time t_c. Then we check if our guess is right. Notice that, although the above rule is arbitrary, any other one would not improve notably our results [2]. Even maximum likelihood estimations, which

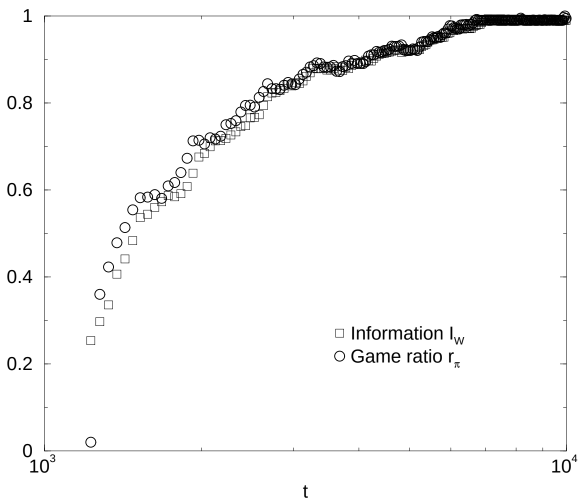

Fig. 2. Information (12) (squares) and percentage r_π of games played by the fittest agent (circles) over time in model (10).

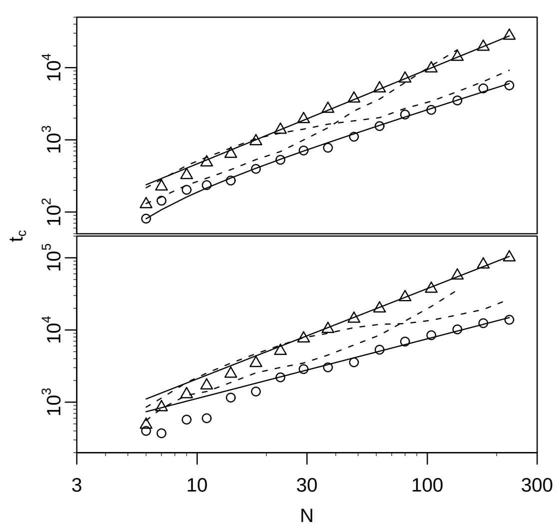

Fig. 3. Average number of games needed to find the best object with probability 0.6 in model (10) with $\pi = 0.60$ in the upper and $\pi = 0.55$ in the lower panel respectively. Symbols report simulation results for a repeated Round Robin tournament (triangles) and for the min-ent method (circles). The dashed lines are for the *ran–fil* with $\eta = 1, 7$, where $\eta = 7$ gives the best performance for large N. In the upper panel the min-ent method is fitted with a linear fit, the robin tournament with a power law fit with exponent 1.5. In the lower panel the fitting exponents are 0.82 respectively 1.25.

are the best ones under our hypothesis, give the same ranking as ours in the case of Round Robins [4,8] yet involving much heavier calculations.

We tested the min-ent method on the model (10). First we verified that it converges, i.e. entropy (11) goes to zero. In figure 2 we show that it is actually the case: the information gain approaches 1,

$$I_{W(t)} = 1 - H\left(\psi | W(t)\right) / H_{max} \to 1, \tag{12}$$

as we employ more resources ($t \to \infty$), and so does the percentage r_π of games played by the fittest agent. Then we compared the performance of the min-ent method with that of the ran-fil method and with repeated RRs; results are shown in figure 3. The total number of comparisons t_c needed to pick the best item with a given probability (0.6 in the figure) seems to scale linearly with the number of items N using the min-ent method, while it is a super-linear power law for RRs. The ran-fil method, although less performing than the min-ent, clearly outperforms Round Robins. This last result is particularly promising, since it is widely applicable to real filters.

Conclusions

We have analyzed different methods of selecting a set of N objects by means of n_c pairwise comparisons. In particular we focused on the amount of resources one has to spend in order to select the fittest. We stated that the best one can obtain with repeated round robins with successive elimination, for a wide variety of probabilistic models, is $n_c \propto N^2 \log N$. Then we introduced two new methods of performing the selection. Both ones give better results than repeated RRs in a worst-case test model: with the min-ent, based on information maximization, we obtained $n_c \propto N$. The ran-fil method has been shown to outperform round robins without requiring any previous knowledge of the underlying model. Its basic principles are to keep an almost constant selecting power at each round and to gradually eliminate losers. We believe they constitute a useful proposal for improving tournament design, with particular reference to ranking methods of Internet Search Engines.

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